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Photon Number Emission in Synchrotron Radiation: Systematics for High-Energy Particles

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Summary. — A recent derivation of an explicit elementary expression for the mean number $\langle N \rangle$ of photons emitted per revolution in synchrotron radiation allows a systematic high-energy analysis leading to the result

$$\langle N \rangle \simeq 5\pi\alpha/\sqrt{3(1-\beta^2)} + a_0\alpha + \pi\alpha\sqrt{1-\beta^2}/10\sqrt{3}$$

where a_0 is a constant, with relative errors of 2.2%, .64%, .017%, in comparison to the well known formula tabulated in the literature of 160%, 82%, 17% for $\beta = 0.8, 0.9, 0.99$, respectively.

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1. – Introduction.

Although many features of synchrotron radiation have been well known for a long time (e.g., [1]), there is certainly room for further improvements and developments. In a recent investigation [2], starting from Schwinger's monumental work on synchrotron radiation [3] many years ago, an expicit expression for the mean number $\langle N \rangle$ of photons emitted per revolution was derived involving a remarkably simple one-dimensional integral. This allowed us to carry out a systematic analysis for high-energy charged particles. Our new expression is given by

$$\langle N \rangle \simeq 5\pi\alpha/\sqrt{3(1-\beta^2)} + a_0\alpha + \pi\alpha\sqrt{1-\beta^2}/10\sqrt{3}$$

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where a_0 is a numerical constant (see (13)). The relative errors are 2.2%, .64%, .017% in comparison to the well known formula tabulated in the literature (see [4]) involving only the first term in the above formula with relative errors of 160%, 82%, 17% for $\beta = 0.8, 0.9, 0.99$, respectively, which in turn has urged us and motivated us to embark in the present investigation.

Our starting point for $\langle N\rangle$ is derived [2] from the Schwinger expression for the power ([3] Eq.III.6, Eq.III.7) giving:

$$\langle N\rangle = \alpha \int_0^\infty \mathrm{d}z \int_{-\infty}^\infty \mathrm{d}x\ \mathrm{e}^{-\mathrm{i}zx}(\beta^2\cos x - 1)\frac{\sin(2z\beta\sin\frac{1}{2}x)}{\beta\sin\frac{1}{2}x} \tag{1}$$

Since the integrand factor in (1), multiplying $\exp(-\mathrm{i}zx)$, is an even function of x, only the real part of the integral in (1) contributes. It is easily verified that $\langle N\rangle = 0$ for $\beta = 0$, as it should be, when integrating over x and z in (1) and by using, in the process, that $\int_0^\infty \mathrm{d}z\ z\int_{-\infty}^\infty \mathrm{d}x\ \mathrm{e}^{-\mathrm{i}zx} = 0$. This latter boundary condition may be explicitly taken into account for the vanishing of $\langle N\rangle$ for $\beta = 0$ by rewriting (1) as

$$\langle N\rangle = \alpha \int_0^\infty \mathrm{d}z \int_{-\infty}^\infty \mathrm{d}x\ \mathrm{e}^{-\mathrm{i}zx}\int_0^\beta \mathrm{d}\rho \left(\cos x\frac{\sin(2\beta z\sin\frac{1}{2}x)}{\sin\frac{1}{2}x} - \frac{2z}{\beta}\left[\cos\left(2z\rho\sin\frac{1}{2}x\right) - 1\right]\right) \tag{2}$$

We first integrate over z, then over ρ and finally make a change of variable $x/2 \to z$ to obtain the remarkably simple expression

$$\langle N\rangle = 2\alpha\beta^2\int_0^\infty \frac{\mathrm{d}z}{z^2}\ \frac{\left(\frac{\sin z}{z}\right)^2 - \cos(2z)}{\left[1 - \beta^2\left(\frac{\sin z}{z}\right)^2\right]} \equiv \alpha f(\beta) \tag{3}$$

There is no question of the existence of the latter integral for all $0 \leqslant \beta < 1$. The integral develops a *singularity* in β for $\beta \to 1$.

2. – Systematic Treatment of $\langle N\rangle$ for High-Energy Particles.

Although the integral expression for $\langle N\rangle$ is simple, the investigation of $\langle N\rangle$ for $\beta \to 1$ is *far from* trivial. For $\beta \to 1$, the integrand in (1) develops a *singularity* for $z \to 0$. By rewriting the integrand of $f(\beta)$ in (3) in a form suitable to study its behaviour for $z \to 0$ makes its investigation for $z \to \infty$ rather difficult. Accordingly, in Appendix A we have provided asymptotic expansions of some basic functions involved in this work. Guided by these expansions, we rewrite the integrand for $f(\beta)$ in (3) explicitly as

$$\frac{2\beta^2}{z^2}\ \frac{\left(\frac{\sin z}{z}\right)^2 - \cos(2z)}{\left[1 - \beta^2\left(\frac{\sin z}{z}\right)^2\right]} = \frac{10\beta}{3(1-\beta^2) + \beta^2 z^2} + \frac{2}{z^2}\ \frac{6\left(\frac{\sin z}{z}\right)^2 - \cos(2z) - 5}{1 - \left(\frac{\sin z}{z}\right)^2} - 2(1-\beta^2)g(\beta, z) \tag{4}$$

where

$$g(\beta,z)=\frac{1}{z^2}\;\frac{\left[\left(\frac{\sin z}{z}\right)^2-\cos(2z)\right]}{\left[1-\beta^2\left(\frac{\sin z}{z}\right)^2\right]\left[1-\left(\frac{\sin z}{z}\right)^2\right]}+\frac{5}{z^2}\;\frac{\left[\beta z^2-3(1+\beta)\right]}{3(1-\beta^2)(1+\beta)+\beta^2(1+\beta)z^2} \tag{5}$$

The essential point to note in (4) is that we have factored out $(1-\beta^2)$ in defining $g(\beta,z)$. The first term in (4) develops a singularity $1/z^2$ near the origin for $\beta\to 1$. The second term in (4) is independent of β and approaches a constant (see Eq.(A.11) for $z\to 0$, and vanishes like $1/z^2$ for $z\to\infty$.

To investigate the contribution of $g(\beta,z)$ to $\langle N\rangle$, we rewrite the former as:

$$g(\beta,z)=\frac{9}{30}\;\frac{\beta}{(1-\beta^2)+\beta^2z^2/3}-\frac{2}{3}\;\frac{(1-\beta^2)\beta}{\left[(1-\beta^2)+\beta^2z^2/3\right]^2}+g_1(\beta,z) \tag{6}$$

where

$$\begin{aligned} g_1(\beta,z)=\frac{1}{z^2}\;&\frac{\left[\left(\frac{\sin z}{z}\right)^2-\cos(2z)\right]}{\left[1-\beta^2\left(\frac{\sin z}{z}\right)^2\right]\left[1-\left(\frac{\sin z}{z}\right)^2\right]}+\left[\frac{5}{3(1+\beta)}-\frac{9}{30}+\frac{2}{3}\;\frac{(1-\beta^2)}{1-\beta^2+\beta^2z^2/3}\right]\\ &\times\frac{\beta}{\left[1-\beta^2+\beta^2z^2/3\right]}-\frac{5}{z^2}\;\frac{1}{1-\beta^2+\beta^2z^2/3}. \end{aligned} \tag{7}$$

In particular, we note that

$$g_1(1,z)=\frac{1}{z^2}\left(\frac{\left(\frac{\sin z}{z}\right)^2-\cos(2z)}{\left[1-\left(\frac{\sin z}{z}\right)^2\right]^2}+\frac{8}{5}-\frac{15}{z^2}\right), \tag{8}$$

and that (see Eq.(A.6))

$$g_1(1,z)\xrightarrow[z\to 0]{}-\frac{101}{600} \tag{9}$$

$$g_1(1,z)\xrightarrow[z\to\infty]{}\mathcal{O}(1/z^2). \tag{10}$$

Using the integrals:

$$\int_0^\infty\frac{\mathrm{d}z}{[3+z^2]}=\frac{\pi}{2\sqrt{3}} \tag{11}$$

$$\int_0^\infty\frac{\mathrm{d}z}{[3+z^2]^2}=\frac{\pi}{12\sqrt{3}} \tag{12}$$

$$2\int_0^\infty \frac{\mathrm{d}z}{z^2}\,\frac{6\left(\frac{\sin z}{z}\right)^2-\cos(2z)-5}{\left[1-\left(\frac{\sin z}{z}\right)^2\right]} \equiv a_0 = -9.55797 \tag{13}$$

we obtain from (3)–(13) the explicit expression

$$f(\beta) = \frac{5\pi}{\sqrt{3(1-\beta^2)}} + a_0 + \frac{\pi}{10\sqrt{3}}\sqrt{1-\beta^2} + \varepsilon(\beta) \tag{14}$$

where

$$\varepsilon(\beta) = -2(1-\beta^2)\int_0^\infty \mathrm{d}z\; g_1(\beta,z) \tag{15}$$

a_0 is given in (13) and $g_1(\beta,z)$ is defined through (7), (8) and (10).

The error term $\varepsilon(\beta)$ is studied in Appendix B giving

$$\varepsilon(\beta) = \mathcal{O}[(1-\beta^2)] \tag{16}$$

for $\beta \to 1$.

Eq.(14) may be also rewritten as

$$f(\beta) = \frac{5\pi}{\sqrt{3(1-\beta^2)}} + a_0 + \varepsilon_0(\beta) \tag{17}$$

where now the error $\varepsilon_0(\beta)$ is from (14) and (16)

$$\varepsilon_0(\beta) = \mathcal{O}(\sqrt{1-\beta^2}) \tag{18}$$

for $\beta \to 1$.

According to Eqs.(14) and (17), we obtain the following representation for $\langle N\rangle$ for high-energy charged particles:

$$\langle N\rangle \simeq \frac{5\pi\alpha}{\sqrt{3(1-\beta^2)}} + a_0\alpha + \frac{\pi\alpha\sqrt{1-\beta^2}}{10\sqrt{3}} \tag{19}$$

where the numerical constant a_0 is given in (13), with relative errors as mentioned in the the Introduction to be compared with the well known tabulated formula [4]. It is important to emphasize that the asymptotic constant a_0 is overwhelmingly large in magnitude and may be easily missed in a non-systematic analysis and is the very important contribution in (19).

Appendix A

We provide asymptotic expansions of some basic functions involved in this work for $z \to 0$. To this end we note the following expansions:

$$\left(\frac{\sin z}{z}\right)^2 - \cos(2z) \cong \frac{5}{3}z^2 - \frac{28}{45}z^4 + \frac{31}{360}z^6 + \cdots \tag{A.1}$$

$$\left(\frac{\sin z}{z}\right)^2 \cong 1 - \frac{z^2}{3} + \frac{2}{45}z^4 - \frac{z^6}{360} + \cdots \tag{A.2}$$

$$\left[1 - \left(\frac{\sin z}{z}\right)^2\right]^2 \cong \frac{z^4}{9}\left(1 - \frac{4}{15}z^2 + \frac{31}{900}z^4\right) + \cdots \tag{A.3}$$

These expressions lead to the following asymptotic expansions for the functions:

$$f_1(z) = \frac{\left(\frac{\sin z}{z}\right)^2 - \cos(2z)}{\left[1 - \left(\frac{\sin z}{z}\right)^2\right]^2} \tag{A.4}$$

$$f_2(z) = \frac{\left(\frac{\sin z}{z}\right)^2 - \cos(2z)}{\left[1 - \left(\frac{\sin z}{z}\right)^2\right]^2}\left(\frac{\sin z}{z}\right)^2, \tag{A.5}$$

$$f_1(z) \xrightarrow[z\to 0]{} \frac{15}{z^2} - \frac{8}{5} - \frac{101}{600}z^2 + \mathcal{O}(z^4) \tag{A.6}$$

$$f_2(z) \xrightarrow[z\to 0]{} \frac{15}{z^2} - \frac{33}{5} + \frac{619}{600}z^2 + \mathcal{O}(z^4) \tag{A.7}$$

and the following asymptotic expansions for the functions:

$$f_3(z) = \frac{6\left(\frac{\sin z}{z}\right)^2 - \cos(2z) - 5}{1 - \left(\frac{\sin z}{z}\right)^2} \tag{A.8}$$

$$f_4(z) = \frac{\left(\frac{\sin z}{z}\right)^2 - \cos(2z)}{\left[1 - \left(\frac{\sin z}{z}\right)^2\right]^2}\left(\frac{\sin z}{z}\right)^2\left[\frac{z^2}{3} + \left(\frac{\sin z}{z}\right)^2 - 1\right] \tag{A.9}$$

(A.10)
$$f_5(z) \equiv \frac{\left(\frac{\sin z}{z}\right)^2 - \cos(2z)}{\left[1-\left(\frac{\sin z}{z}\right)^2\right]^2} \left(\frac{\sin z}{z}\right)^2 \left[\frac{z^2}{3} + \left(\frac{\sin z}{z}\right)^2 - 1\right]^2,$$

(A.11)
$$f_3(z) \xrightarrow[z\to 0]{} -\frac{6}{5}z^2 + \mathcal{O}(z^4)$$

(A.12)
$$f_4(z) \xrightarrow[z\to 0]{} \frac{2}{3}z^2 - \frac{67}{200}z^4 + \mathcal{O}(z^6)$$

(A.13)
$$f_5(z) \xrightarrow[z\to 0]{} \frac{4}{135}z^6 + \mathcal{O}(z^8)$$

Appendix B

In this appendix we investigate the nature of the error term in (14). To this end, we rewrite the integrand $g_1(\beta, z)$ defined in (7) as

(B.1)
$$g_1(\beta, z) = [g_1(\beta, z) - g_1(1, z)] + g_1(1, z),$$

(B.2)
$$\begin{aligned} g_1(\beta, z) - g_1(1, z) = &-\frac{(1-\beta^2)}{z^2} \frac{f_2(z)}{\left[1-\beta^2\left(\frac{\sin z}{z}\right)^2\right]} + \frac{2}{3}\frac{(1-\beta^2)\beta}{[1-\beta^2+\beta^2 z^2/3]^2} \\ &+ \frac{57}{120}\frac{(1-\beta^2)\beta}{[1-\beta^2+\beta^2 z^2/3]} + \frac{(1-\beta^2)^2\beta(107+57\beta)}{120(1+\beta)^3[1-\beta^2+\beta^2 z^2/3]} \\ &- \frac{33}{5z^2}\frac{(1-\beta^2)}{[1-\beta^2+\beta^2 z^2/3]} + \frac{15}{z^4}\frac{(1-\beta^2)}{[1-\beta^2+\beta^2 z^2/3]} \end{aligned}$$

and where $f_2(z)$ is defined in (A.5).

Upon using the identity

(B.3)
$$\begin{aligned} \frac{1}{\left[1-\beta^2\left(\frac{\sin z}{z}\right)^2\right]} = &\frac{\beta^6\left[\frac{z^2}{3}+\left(\frac{\sin z}{z}\right)^2-1\right]^3}{[1-\beta^2+\beta^2 z^2/3]^3\left[1-\beta^2\left(\frac{\sin z}{z}\right)^2\right]} + \frac{\beta^4\left[\frac{z^2}{3}+\left(\frac{\sin z}{z}\right)^2-1\right]^2}{[1-\beta^2+\beta^2 z^2/3]^3} \\ &+ \frac{\beta^2\left[\frac{z^2}{3}+\left(\frac{\sin z}{z}\right)^2-1\right]}{[1-\beta^2+\beta^2 z^2/3]^2} + \frac{1}{[1-\beta^2+\beta^2 z^2/3]} \end{aligned}$$

we may rewrite (B.2) as

$$g_1(\beta,z)-g_1(1,z)=\sum_{i=1}^{5}F_i(\beta,z) \tag{B.4}$$

where

$$F_1(\beta,z)=-\frac{(1-\beta^2)\beta^6 f_5(z)\left[\frac{z^2}{3}+\left(\frac{\sin z}{z}\right)^2-1\right]}{z^2\left[1-\beta^2\left(\frac{\sin z}{z}\right)^2\right][1-\beta^2+\beta^2z^2/3]^3} \tag{B.5}$$

$$F_2(\beta,z)=-\frac{(1-\beta^2)\beta^4 f_5(z)}{z^2[1-\beta^2+\beta^2z^2/3]^3} \tag{B.6}$$

$$F_3(\beta,z)=-\frac{(1-\beta^2)\beta^4 f_4(z)}{z^2[1-\beta^2+\beta^2z^2/3]^2}+\frac{2}{3}\frac{(1-\beta^2)}{[1-\beta^2+\beta^2z^2/3]^2} \tag{B.7}$$

$$F_4(\beta,z)=-\frac{(1-\beta^2)(f_2(z)+33/5-15/z^2)}{z^2[1-\beta^2+\beta^2z^2/3]} \tag{B.8}$$

$$F_5(\beta,z)=\frac{57}{120}\frac{(1-\beta^2)\beta}{[1-\beta^2+\beta^2z^2/3]}+\frac{(1-\beta^2)^2\beta(107+57\beta)}{120(1+\beta)^3[1-\beta^2+\beta^2z^2/3]} \tag{B.9}$$

and $f_2(z)$, $f_4(z)$, $f_5(z)$ are defined, respectively, in (A.5), (A.9), (A.10). From the asymptotic expansions of the latter functions in (A.7), (A.12), (A.13), respectively, we infer that $g_1(\beta,z)-g_1(1,z)$ gives a $\mathcal{O}\left((1-\beta^2)^{3/2}\right)$ contribution to $\varepsilon(\beta)$. On the other hand, from (8)–(10), we infer that $g_1(1,z)$ in (B.1) will give a $\mathcal{O}\left((1-\beta^2)\right)$ contribution to $\varepsilon(\beta)$. All told we infer that $\varepsilon(\beta)=\mathcal{O}\left((1-\beta^2)\right)$ for $\beta\to 1$.

REFERENCES

[1] H. C. Pollock, "The Discovery of Synchrotron Radiation", *Am. J. Phys.* **51** (3), 278 (1983), and references therein;
V. A. Bordovitsyn (Ed.), *Synchrotron Radiation and its Development*, World Scientific (1999), Singapore, for most recent developments, and the very extensive references therein.
[2] E. B. Manoukian and N. Jearnkulprasert, *Phys. Lett.* **A268**, 35 (2000).
[3] J. Schwinger, *Phys. Rev.* **75**, 1912 (1949).
[4] Review of Particle Physics, *Phys. Rev.* **D54** (Part I), p. 75 (1996);
Eur. Phys. J. **C3**, p. 79 (1998); *ibid.* **C15**, p. 83 (2000); *Phys. Rev.* **D66** (Part I), p. 010001-87 (2002).